MTH 161 - Fall 2013

Lecture 6.

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## Continuity

A function is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

So this definition automatically implies three things for f to be continuous at a

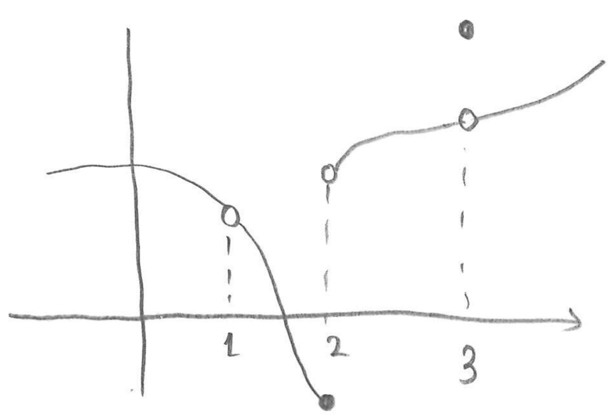
1) $f(a)$ is defined (i.e. a is in the domain of f)

2) $\lim_{x \to a} f(x)$ exists.

3) $\lim_{x \to a} f(x) = f(a)$

If f is defined near a, we say f is discontinuous at a, if f is not continuous at a.

Ex




Where is f discontinuous?

$x = 1$ because $1 \notin$ Domain of f

$x = 2$, $\lim_{x \to 2} f(x)$ DNE

$x = 3$, $\lim_{x \to 3} f(x) \neq f(3)$

- A function $f$ is continuous from the right at a number $a$ if

$$\lim_{x \to a^+} f(x) = f(a)$$

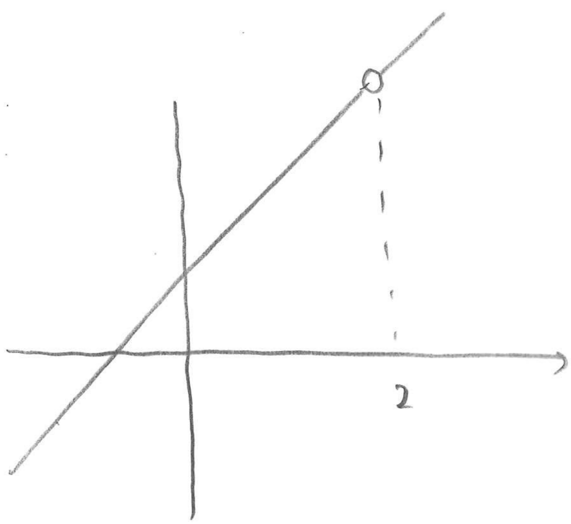
and $f$ is continuous from the left at a number $a$ if

$$\lim_{x \to a^-} f(x) = f(a)$$

Ex

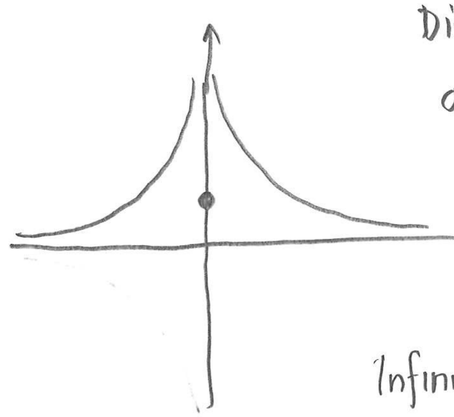
$$f(x) = \frac{x^2 - x - 2}{x - 2} = \frac{(x-2)(x+1)}{(x-2)}$$

Discontinuous at 2.




Removable discontinuity

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

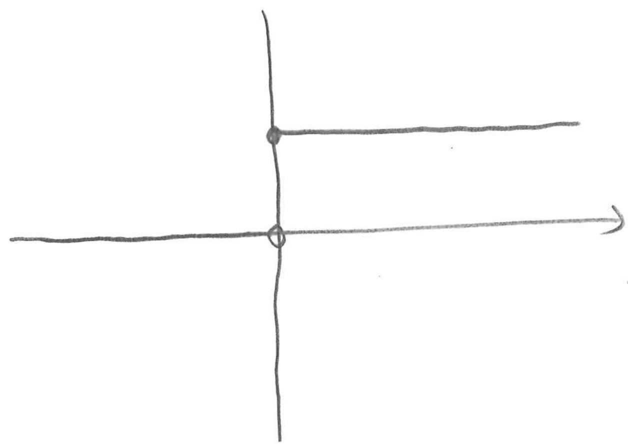


Discontinuous at $x = 0$.

Infinite discontinuity

$$H(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0 \end{cases}$$

Jump discontinuity.



Discontinuous at $x = 0$

Defn A function is continuous on an interval if it is continuous on an interval if it continuous at every number in the interval (If $f$ is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the interval from the left)

~~So is $H(t)$ continuous on $[0, 1]$~~ ?

Ex

Show that the function $H(t)$ is continuous on $[0,1]$

So first we need to consider on $(0,1)$.

For each pt ~~here w~~ $a \in (0,1)$

$$\lim_{t \to a} H(t) = 1 \text{ and } H(a) = 1 .$$

$$\lim_{t \to 0^+} H(t) = 1 \text{ and } H(0) = 1$$

$$\lim_{t \to 1^-} H(t) = 1 \text{ and } H(1) = 1$$

$\boxtimes$

Thm If $f$ and $g$ are continuous at $a$ and $c$ is a constant, then the following are also continuous at $a$.

1) $f+g$ 2) $f-g$ 3) $fg$ 4) $cf$ 5) $\frac{f}{g}$ if $g(a) \neq 0$.

Let's prove the first one

f and g are continuous at a means

that $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$

Now,

$$\lim_{x \to a} (f+g)(x) = \lim_{x \to a} \left[ f(x) + g(x) \right] \longrightarrow \text{Def}^{n}$$

$$= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \longrightarrow \text{Limit Law}$$

$$= f(a) + g(a) = (f+g)(a)$$

Hence $f+g$ is continuous at a.

Thm

a) A polynomial is continuous everywhere, continuous on $(-\infty, \infty)$

b) A rational function is continuous wherever it is defined, that is,

it is continuous on it's domain.

These are exactly the as direct substitution property

Thm Polynomials, rational functions, root function, trigonometric

functions are continuous on their domain

Ex On which interval is the function continuous ?

a) $f(x) = x^{100} - 2x^{49} + 25 \longrightarrow (-\infty, \infty)$

b) $g(x) = \dfrac{x^2 + 3x + 14}{x^2 - 1} \rightarrow x \neq \pm 1$.

c) $h(x) = \sqrt{x} + \dfrac{x+1}{x-1}$, $x \geq 0 \cap x \neq 1$.

$\Downarrow$

$[0, 1) \cup (1, \infty)$

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Thm If $f$ is continuous at $b$ and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(b) .$$

In other words, $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$

Thm If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at $a$.

- Continuous function of a continuous function is continuous .

Example

$h(x) = \sin(x^2)$

$h(x) = f(g(x))$ , where $g(x) = x^2$, $f(x) = \sin x$

$g$ is continuous on $\mathbb{R}$, since it is poly

$h$ is continuous on $\mathbb{R}$, since

Then $h = f \circ g$ is continuous on $\mathbb{R}$

Ex $F(x) = \dfrac{1}{\sqrt{x^2+7}-4}$

Then we have

$K(x) = x^2+7$, $h(x) = \sqrt{x}$, $g(x) = x-4$, $f(x) = \frac{1}{x}$

Each of these is continuous on it's domain, so the function

is continuous on it's domain

$\{x \mid \sqrt{x^2+7} - 4 \neq 0\}$ and $\underbrace{x^2+7 \geq 0}_{\text{satisfied by all real numbers}}$.

$\sqrt{x^2+7} - 4 = 0$

$\sqrt{x^2+7} = 4$

$x^2+7 = 16$

$x^2 = 9$

$x = \pm 3$

So, continuous on

$(-\infty,-3) \cup (-3,3) \cup (3,\infty)$

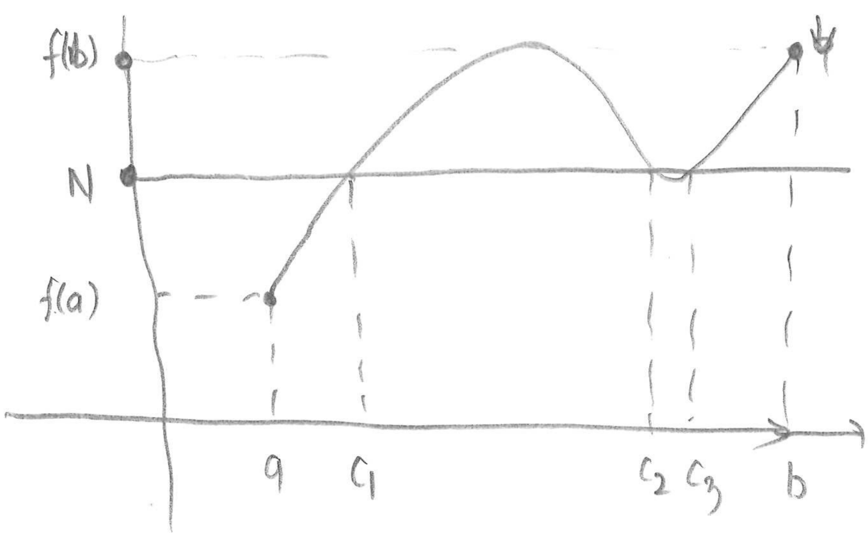
## INTERMEDIATE VALUE ~~OF~~ THM

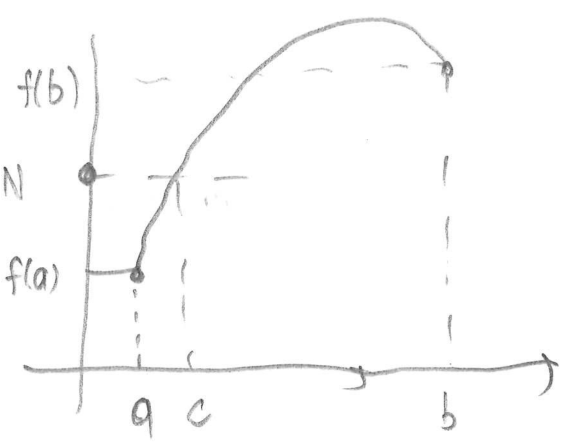
Spse $f$ is continuous on the closed interval $[a,b]$ and let $N$ be any number betn $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c$ in $(a,b)$ s.t $f(c) = N$.

IDEA

What it really says, a continuous function takes on every value between $f(a)$ and $f(b)$.

Graphically



Continuous as no hole or no break.

Ex  Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0 \quad \text{betn } 1 \,\&\, 2 \text{ ??}$$

Soln  let $f(x) = 4x^3 - 6x^2 + 3x - 2$

We are looking for a soln of the given eqn i.e. a number c betn 1 and 2 s.t $f(c) = 0$. So we are going to use $a = 1, b = 2, N = 0$ in Thm.

$$f(1) = 4(1) - 6(1) + 3 - 2 = -1 < 0$$

$$f(2) = 4(2)^3 - 6(2)^2 + 3.2 - 2 = 32 - 24 + 6 - 2 = 12 > 0.$$

Now since $f$ is a polynomial, it is continuous everywhere, and hence on $[1, 2]$

Now we have $f(1) < 0 < f(2)$, and hence $N = 0$, there is a number c between 1 and 2.

Hence, the eq$^n$ $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in interval $(1, 2)$